AUTOMATIC CLOSURE OF INVARIANT LINEAR MANIFOLDS FOR OPERATOR ALGEBRAS

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Abstract. Kadison's transitivity theorem implies that, for irreducible representations of C*-algebras, every invariant linear manifold is closed. It is known that CSL algebras have this property if, and only if, the lattice is hyperatomic (every projection is generated by a finite number of atoms). We show several other conditions are equivalent, including the condition that every invariant linear manifold is singly generated.

We show that two families of norm closed operator algebras have this property. First, let $\mathcal{L}$ be a CSL and suppose $\mathcal{A}$ is a norm closed algebra which is weakly dense in $\operatorname{Alg}\mathcal{L}$ and is a bimodule over the (not necessarily closed) algebra generated by the atoms of $\mathcal{L}$. If $\mathcal{L}$ is hyperatomic and the compression of $\mathcal{A}$ to each atom of $\mathcal{L}$ is a C*-algebra, then every linear manifold invariant under $\mathcal{A}$ is closed. Secondly, if $\mathcal{A}$ is the image of a strongly maximal triangular AF algebra under a multiplicity free nest representation, where the nest has order type $-\mathbb{N}$, then every linear manifold invariant under $\mathcal{A}$ is closed and is singly generated.

The Kadison Transitivity Theorem [15] states, in part, that if π is an irreducible representation of a C*-algebra $\mathcal{C}$ acting on a Hilbert space $\mathcal{H}$, then each linear manifold invariant under $\pi(\mathcal{C})$ is closed. What other representations also have the property that every invariant linear manifold is closed? It is not difficult to extend Kadison's result to show that if π is a representation of $\mathcal{C}$ on $\mathcal{H}$ then every invariant linear manifold for $\pi(\mathcal{C})$ is closed if, and only if, the commutant of the image, $\pi(\mathcal{C})'$, is a finite dimensional C*-algebra (we outline an argument below). This condition is in turn equivalent to $\pi(\mathcal{C})$ being the finite direct sum of irreducible C*-algebras. The summands are unitarily inequivalent if, and only if, $\pi(\mathcal{C})'$ is abelian. Thus, if π is a multiplicity free representation, every invariant linear manifold for $\pi(\mathcal{C})$ is closed if, and only if, the lattice of invariant closed subspaces $\operatorname{Lat}\pi(\mathcal{C})$ is a finite Boolean algebra. In the language introduced below, this says $\operatorname{Lat}\pi(\mathcal{C})$ is a hyperatomic lattice.

The main purpose of this note is to give analogous results for operator algebras which are not C*-algebras and which have proper closed invariant subspaces. Suppose that $\mathcal{C}$ is a C*-algebra and that $\mathcal{D} \subseteq \mathcal{C}$ is a maximal abelian $*$-subalgebra of $\mathcal{C}$. We are interested in representations of intermediate algebras $\mathcal{A}$ (so that $\mathcal{D} \subseteq \mathcal{A} \subseteq \mathcal{C}$). In this context, a *representation of* $\mathcal{A}$ will always be the restriction of a $*$-representation of $\mathcal{C}$ to $\mathcal{A}$. Such algebras and their representations have been considered by numerous authors, including Arveson [1, 2, 3], Muhly, Qiu and Solel [18], Peters, Poon and Wagner [23], and Power [26]. In particular, representations of such algebras have been studied by Orr and Peters [21] and by Muhly and Solel [20]. One motivation for considering such representations is that, under reasonable

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hypotheses, (general) representations of $\mathcal{A}$ can often be dilated to $*$-representations of $\mathcal{C}$; see, for example, [1, 2, 19, 8].

An important special case is when $\mathcal{C} = \mathcal{B}(\mathcal{H})$, $\mathcal{A}$ is a CSL algebra contained in $\mathcal{C}$, and π is the identity representation. Foiaş [9, 10] determined when every invariant operator range for a nest algebra is closed. Davidson described invariant operator ranges for reflexive algebras in [4]. These results were extended by the second author to invariant linear manifolds of CSL algebras in [13], where it is shown that every invariant linear manifold for a CSL algebra is closed if, and only if, the invariant subspace lattice is hyperatomic.

The closely related notions of strictly irreducible and topologically irreducible representations have been studied for Banach algebras; see, for example, [6]. Also relevant is the transitive algebra problem, which asks if an unital operator algebra $\mathcal{A}$ with $\operatorname{Lat}\mathcal{A} = \{0, I\}$ must be weakly dense in $\mathcal{B}(\mathcal{H})$? An affirmative answer would, of course, also settle the invariant subspace problem. It is known that an algebraically transitive subalgebra of $\mathcal{B}(\mathcal{H})$ must be weakly dense in $\mathcal{B}(\mathcal{H})$; see [28, Chapter 8]. Thus, showing that topological transitivity implies algebraic transitivity for norm closed operator algebras would also settle the transitive algebra problem.

Returning to our context, let $\mathcal{D} \subseteq \mathcal{A} \subseteq \mathcal{C}$ as above and let π be a $*$-representation of $\mathcal{C}$ such that every invariant linear manifold for $\pi(\mathcal{A})$ is closed. We wish to observe that in many situations, (for example, when $\pi(\mathcal{D})''$ is a masa in $\mathcal{B}(\mathcal{H})$), $\pi(\mathcal{A})$ is σ-weakly dense in a CSL algebra. To see this, note that,

$$\pi(\mathcal{D})'' = \operatorname{Alg}\operatorname{Lat}(\pi(\mathcal{D})) \subseteq \operatorname{Alg}\operatorname{Lat}(\pi(\mathcal{A})).$$

When $\pi(\mathcal{D})''$ is a masa in $\mathcal{B}(\mathcal{H})$ or, more generally, $\operatorname{Alg}\operatorname{Lat}(\pi(\mathcal{A}))$ contains a masa, then $\operatorname{Alg}\operatorname{Lat}(\pi(\mathcal{A}))$ is a CSL algebra with invariant subspace lattice $\mathcal{L} := \operatorname{Lat}(\pi(\mathcal{A}))$. Since every invariant manifold for $\pi(\mathcal{A})$ is closed, so is every invariant manifold for $\operatorname{Alg}\mathcal{L}$. Therefore, $\mathcal{L}$ is hyperatomic and, in particular, is also atomic. By [3, Theorem 2.2.11], $\operatorname{Alg}\mathcal{L}$ is synthetic, and then [3, Corollary 2 of Theorem 2.1.5] shows that $\pi(\mathcal{A})$ is σ-weakly dense in $\operatorname{Alg}\mathcal{L}$.

For many of the examples appearing in [21], $\pi(\mathcal{D})''$ is a masa in $\mathcal{B}(\mathcal{H})$, so that $\mathcal{L} := \operatorname{Lat}(\pi(\mathcal{A}))$ is a commutative subspace lattice. Other examples from [21] show that even when $\pi(\mathcal{D})''$ is not a masa, $\operatorname{Alg}\operatorname{Lat}(\pi(\mathcal{A}))$ may still contain a masa, and thus $\operatorname{Alg}\operatorname{Lat}(\pi(\mathcal{A}))$ is again a CSL algebra with lattice $\operatorname{Lat}(\pi(\mathcal{A}))$.

For these reasons, we shall always assume that $\pi(\mathcal{A})$ is contained in the CSL algebra $\operatorname{Alg}\mathcal{L}$, where $\mathcal{L} = \operatorname{Lat}(\pi(\mathcal{A}))$.

In section one, we obtain several conditions on a CSL algebra which are equivalent to the condition that every invariant linear manifold is closed, and then give an automatic closure result for norm closed operator algebras which are weakly dense in a CSL-algebra. In the second section, we turn to a specific family of norm closed operator algebras, those arising as representations of triangular AF (TAF) algebras.

We now turn to a few matters of notation. All Hilbert spaces in this paper will be separable. The symbol $\mathcal{L}$ always denotes a CSL, that is, a strongly closed lattice of mutually commuting projections containing 0 and I.

Given a (not necessarily closed) operator algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, we let $\operatorname{Man}\mathcal{A}$ denote the set of all linear manifolds of $\mathcal{H}$ invariant under $\mathcal{A}$. We use $\operatorname{Lat}\mathcal{A}$ for the set of all *closed* subspaces of $\mathcal{H}$ which are invariant under $\mathcal{A}$. Clearly $\operatorname{Lat}\mathcal{A} \subseteq \operatorname{Man}\mathcal{A}$ and the closure of every element of $\operatorname{Man}\mathcal{A}$ belongs to $\operatorname{Lat}\mathcal{A}$. Given a vector $x \in \mathcal{H}$, we use $\{\mathcal{A}x\}$ for $\{Ax \mid A \in \mathcal{A}\}$

and $[\mathcal{A}x]$ for the closure of $\{\mathcal{A}x\}$. For $\mathcal{A}$ unital, these are the smallest elements of $\operatorname{Man}\mathcal{A}$ and $\operatorname{Lat}\mathcal{A}$ respectively containing x.

A vector $x \in \mathcal{H}$ is called *closed for* $\mathcal{A}$ if $\{\mathcal{A}x\} = [\mathcal{A}x]$. The terminology is justified when one views the vector x as inducing a map $A \mapsto Ax$ from $\mathcal{A}$ into $\mathcal{H}$: a vector is closed when the associated map has closed range. (The notion of a closed vector generalizes the concept of a strictly cyclic vector for an operator algebra, found in [11]: recall that a vector $x \in \mathcal{H}$ is strictly cyclic for $\mathcal{A}$ if $\mathcal{H} = \{\mathcal{A}x\}$.)

An element $M \in \operatorname{Lat}\mathcal{A}$ is *cyclic* if there exists a vector $x \in \mathcal{H}$ such that $M = [\mathcal{A}x]$. To avoid further overloading the word cyclic, we call an invariant linear manifold M *singly generated* if there is a vector $x \in \mathcal{H}$ with $M = \{\mathcal{A}x\}$, whether M is closed or not.

Finally, we outline the argument showing that every invariant linear manifold for π, a $*$-representation of a C*-algebra $\mathcal{C}$, is closed if, and only if, $\pi(\mathcal{C})'$ is finite dimensional. This is presumably known but we have not found a convenient reference.

One direction is trivial: if $\pi(\mathcal{C})'$ is infinite dimensional, it contains an infinite chain of projections $P_1 < P_2 < \dots$. Then $\mathcal{M} = \cup_{n=1}^{\infty} P_n\mathcal{H}$ is a non-closed invariant manifold for $\pi(\mathcal{C})$.

For the converse, observe that the finite dimensionality of $\pi(\mathcal{C})'$ implies that that we can decompose π as a finite direct sum of irreducible representations, say $\oplus\pi_i$, acting on $\oplus\mathcal{H}_i$. If $\mathcal{M}$ is an invariant linear manifold for $\pi(\mathcal{C})$, then, after a possible rearrangement of the order of the summands, we can express the elements of $\mathcal{M}$ as

$$\big(h_1, \dots , h_k, L_{k+1}(h_1, \dots , h_k), \dots , L_n(h_1, \dots , h_k)\big)$$

where each h_i is an arbitrary element of $\mathcal{H}_i$ and each L_i is a linear transformation from $\oplus_{j=1}^{k}\mathcal{H}_j \to \mathcal{H}_i$. If L is the restriction of L_i to some $\mathcal{H}_j$, $j \le k$, then L intertwines the actions of π_j and π_i.

We claim that L is a scalar multiple of a unitary. Fix a unit vector $x \in \mathcal{H}_j$. For each unit vector $v \in \mathcal{H}_j$, by Kadison's transitivity theorem, there is a unitary $U \in \mathcal{C}$ so that $\pi_j(U)x = v$. As $Lv = L\pi_j(U)x = \pi_i(U)Lx$, we can conclude that $\|Lv\| = \|Lx\|$ for all unit vectors v. Thus, $\|L\| = \|Lx\|$ for each unit vector x and so L is a scalar multiple of an isometry. The transitivity of π_i implies that if $L \neq 0$ then L is onto, so L is a scalar multiple of a unitary, as claimed. Thus we may write each L_i as a linear combination of unitary operators. It follows that $\mathcal{M}$ is a closed subspace of $\mathcal{H}$.

1. CSL Algebras

Definition 1. A projection P in $\mathcal{L}$ is *hyperatomic* if there are finitely many atoms $A_1, \dots , A_k$ from $\mathcal{L}$ such that $P = E(A_1, \dots , A_k)$, the smallest projection in $\mathcal{L}$ containing $A_1, \dots , A_k$. We say that P is *generated* by $A_1, \dots , A_k$ if $P = E(A_1, \dots , A_k)$. If each non-zero projection in $\mathcal{L}$ is hyperatomic, we say that the lattice is *hyperatomic.*

Remark. A projection P is hyperatomic if, and only if, each ascending sequence $F_1 \le F_2 \le F_3 \le \dots$ with $P = \vee F_n$, $F_n \in \mathcal{L}$, is eventually constant. Indeed, assume that P is generated by the atoms $A_1, \dots , A_k$. Given an increasing sequence F_n of elements of $\mathcal{L}$ with $P = \vee F_n$, there is, for each $j = 1, \dots , n$, a projection F_{n_j} such that $A_j \le F_{n_j}$. If $m \ge \max\{n_j \mid j = 1, \dots , k\}$ then $P = E(A_1, \dots , A_k) \le F_m \le P$; hence $F_m = P$, for all large m.

On the other hand, assume the ascending chain condition. Let $\mathcal{S}$ be the set of atoms contained in P. If $E(\mathcal{S}) < P$, then $P - E(\mathcal{S})$ contains no atoms, i.e., no non-zero subinterval which is minimal. In this case it is easy to construct a strictly increasing sequence $F_1 <$

$F_2 < \dots$ in $\mathcal{L}$ such that $P = \vee F_n$, contradicting the ascending chain condition. Thus we may assume that P is generated by the atoms which it contains. If P is not generated by finitely many atoms, let $A_1, A_2, \dots$ be an infinite sequence of atoms which generates P. Then $E(A_1) \leq E(A_1, A_2) \leq \dots \leq E(A_1, \dots, A_k) < P$, for all k, and $P = \vee_k E(A_1, A_2, \dots, A_k)$, again contradicting the ascending chain condition. Thus we may conclude that P is generated by finitely many atoms; i.e., P is hyperatomic.

The version of this remark appropriate to the whole lattice appeared in [13].

For x and y in $\mathcal{H}$, xy^* denotes the rank one operator on $\mathcal{H}$ given by $z \mapsto \langle z, y \rangle\, x$. Also, if $P \in \mathcal{L}$, then P_- denotes $\bigvee\{L \in \mathcal{L} : L \not\geq P\}$.

Lemma 2. *Let A be an atom from $\mathcal{L}$, let x be a non-zero vector in A, let $y \in E(A)$ and put $T := \|x\|^{-2}\; yx^*$. Then $T \in \operatorname{Alg}\mathcal{L}$, $Tx = y$ and $TA = T$.*

Proof. Since $x \in A$, we clearly have $TA = T$. Notice that if $L \in \mathcal{L}$ satisfies $L \not\geq E(A)$, then $AL = 0$; for otherwise $A \leq L$, whence $E(A) \leq L$. Therefore $AE(A)_- = 0$, so $x \in (E(A)_-)^\perp$. Thus, from [17], we see that $T \in \operatorname{Alg}\mathcal{L}$. $\square$

Proposition 3. *Let $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$ and let $x \in \mathcal{H}$. The following are equivalent:*

1. *x is a closed vector for $\operatorname{Alg}\mathcal{L}$, i.e., $\{\operatorname{Alg}\mathcal{L}x\}$ is closed.*
2. *The projection onto $[\operatorname{Alg}\mathcal{L}x]$ is hyperatomic.*

Proof. Let P be the orthogonal projection onto $[\operatorname{Alg}\mathcal{L}x]$. First assume that P is not hyperatomic. Let $F_1 < F_2 < \dots$ be a strictly ascending sequence of projections in $\mathcal{L}$ such that $P = \vee F_n$. Let a_n be a sequence of positive real numbers such that

$$\sum_{n=1}^{\infty} n^2 a_n^2 < \infty. \tag{1}$$

Let k_n be a sequence of positive integers such that for all $n \in \mathbb{N}$,

$$k_{n+1} \geq k_n + 2, \quad \text{and}$$
$$\|(P - F_{k_n})x\| = \|F_{k_n}^\perp x\| \leq a_n.$$

For each n, let $y_n \in (F_{k_n+1} - F_{k_n})\mathcal{H}$ be a vector with $\|y_n\| = na_n$. By (1), the sum $\sum_{n=1}^\infty y_n$ converges to an element $y \in P\mathcal{H}$.

For every $n \in \mathbb{N}$, we have

$$\|F_{k_n}^\perp y\| \geq \|(F_{k_n+1} - F_{k_n})y\| = \|y_n\| = na_n \quad \text{and} \quad \|F_{k_n}^\perp x\| \leq a_n.$$

Hence for all n,

$$\frac{\|F_{k_n}^\perp y\|}{\|F_{k_n}^\perp x\|} \geq \frac{na_n}{a_n} = n.$$

It follows from [12] that $y \neq Tx$ for any $T \in \operatorname{Alg}\mathcal{L}$; i.e., $y \notin \{\operatorname{Alg}\mathcal{L}x\}$. Thus $\{\operatorname{Alg}\mathcal{L}x\} \neq [\operatorname{Alg}\mathcal{L}x]$, so x is not a closed vector.

Now suppose that P is hyperatomic. Let $A_1, \dots, A_n$ be a finite set of atoms of $\mathcal{L}$ such that

$$P = E(A_1, \dots, A_n) = \bigwedge\{F \in \mathcal{L} \mid A_j \leq F, j = 1, \dots, n\}.$$

By deleting some atoms, if necessary, we may assume that $A_1, \dots, A_n$ is a minimal set which generates P. Thus, if $\mathcal{S}$ is any proper subset of $\{A_1, \dots, A_n\}$, then $E(\mathcal{S}) < P$.

Let $x_k = A_k x$, $k = 1, \dots, n$. Note that $x_k \neq 0$, for each k. (Otherwise, we have $\overline{\{\mathrm{Alg}\,\mathcal{L}x\}} \subsetneq P\mathcal{H} = [\mathrm{Alg}\,\mathcal{L}x]$ and x is in $E(A_1, \dots, A_{k-1}, A_{k+1}, \dots, A_n)\mathcal{H}$, a contradiction.) Now let y be any vector in $[\mathrm{Alg}\,\mathcal{L}x] = P\mathcal{H}$. Then there exist vectors $y_1, \dots, y_n$ (not necessarily unique) such that $y_k \in E(A_k)\mathcal{H}$, for each k, and $y = y_1 + \cdots + y_n$. By Lemma 2, there is, for each k, an element $T_k \in \mathrm{Alg}\,\mathcal{L}$ such that $T_k x_k = y_k$ and $T_k = T_k A_k$.

Let $T = T_1 + \cdots + T_k$. Clearly, $T_k x_j = 0$ whenever $k \neq j$. So $Tx = \sum_{k=1}^{n} T_k x_k = \sum_{k=1}^{n} y_k = y$. This shows that $[\mathrm{Alg}\,\mathcal{L}x] \subseteq \{\mathrm{Alg}\,\mathcal{L}x\}$, and it follows that x is a closed vector for $\mathrm{Alg}\,\mathcal{L}$. □

A von Neumann algebra $\mathfrak{M}$ is also a CSL algebra precisely when $\mathfrak{M}'$ is abelian. In this sense, one can view CSL algebras as generalizations of the von Neumann algebras with abelian commutant. The discussion in the introduction shows that every invariant manifold for a von Neumann algebra $\mathfrak{M}$ with abelian commutant is closed exactly when $\mathfrak{M}$ is finite dimensional, or equivalently, when $\mathrm{Lat}(\mathfrak{M})$ is hyperatomic. The next result, Theorem 4, generalizes this characterization to the class of all CSL algebras.

We also remark that Theorem 4 extends work of Froelich in [11]. Motivated by operator theory, Froelich introduced the notions of strictly cyclic operator algebras (those for which there is $x \in \mathcal{H}$ with $\{\mathcal{A}x\} = \mathcal{H}$) and of strongly strictly cyclic operator algebras (those for which the compression of $\mathcal{A}$ to each invariant projection is strictly cyclic). He showed that strict cyclicity is equivalent to the ascending chain condition for the identity and the analogous result for strong strict cyclicity, essentially $1 \Leftrightarrow 4$ in Theorem 4.

Theorem 4. *Let $\mathcal{L}$ be a commutative subspace lattice. The following statements are equivalent.*

1. *$\mathcal{L}$ is a hyperatomic CSL.*
2. *Every invariant manifold for* $\mathrm{Alg}\,\mathcal{L}$ *is closed, i.e.,* $\mathrm{Man}(\mathrm{Alg}\,\mathcal{L}) = \mathcal{L}$.
3. *Every singly generated invariant manifold for* $\mathrm{Alg}\,\mathcal{L}$ *is closed.*
4. *Every element of $\mathcal{L}$ is singly generated; that is, for $P \in \mathcal{L}$ there exists a vector $x \in P\mathcal{H}$ such that* $\{\mathrm{Alg}\,\mathcal{L}x\} = P\mathcal{H}$.
5. *Every invariant manifold for* $\mathrm{Alg}\,\mathcal{L}$ *is singly generated.*

Proof. ($1 \Leftrightarrow 2$) This is proved in [13].

($2 \Rightarrow 3$) Obvious.

($3 \Rightarrow 4$) If P is any element of $\mathcal{L}$ then, since $\mathcal{H}$ is separable, there is a vector $x \in \mathcal{H}$ such that P is the projection onto $[\mathrm{Alg}\,\mathcal{L}x]$. But our hypothesis is that $\{\mathrm{Alg}\,\mathcal{L}x\}$ is already closed, so $\{\mathrm{Alg}\,\mathcal{L}x\} = P\mathcal{H}$.

($4 \Rightarrow 1$) Given $P \in \mathcal{L}$ we may find $x \in \mathcal{H}$ such that $\{\mathrm{Alg}\,\mathcal{L}x\} = P\mathcal{H}$, thus x is a closed vector. By Proposition 3, P is a hyperatomic projection. Since P is arbitrary, every projection is hyperatomic and so $\mathcal{L}$ is hyperatomic.

($5 \Rightarrow 4$) Obvious.

($2 \Rightarrow 5$) If $\mathcal{M}$ is any invariant linear manifold, then $\mathcal{M}$ is closed by hypothesis, so by the equivalence of (2) and (4), we see that there exists a vector x such that $\mathcal{M} = \{\mathrm{Alg}\,\mathcal{L}x\}$. □

We next turn our attention to operator algebras $\mathcal{A}$ which are not weakly closed, but are subalgebras of CSL algebras. Theorem 4 shows that a necessary condition for $\mathrm{Man}(\mathcal{A})$ to

coincide with $\operatorname{Lat}(\mathcal{A})$ is that $\mathcal{A}$ be a subalgebra contained inside the algebra of a hyperatomic CSL, so we will restrict our attention to this setting.

Definition 5. Let $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{C} \subseteq \mathcal{B}(\mathcal{H})$ be operator algebras. We will say that $\mathcal{B}$ is *$\mathcal{C}$-transitive* if $\{\mathcal{B}x\} = \{\mathcal{C}x\}$ for every $x \in \mathcal{H}$. Our primary interest is when $\mathcal{B} \subseteq \mathcal{C}$. Notice that when $\mathcal{C} = \mathcal{B}(\mathcal{H})$, then the statement that $\mathcal{B}$ is $\mathcal{C}$-transitive is simply the statement that $\mathcal{B}$ is a transitive operator algebra.

The next proposition shows how $\operatorname{Alg}\mathcal{L}$-transitivity, closed vectors, and automatic closure of invariant manifolds for an algebra $\mathcal{A} \subseteq \operatorname{Alg}\mathcal{L}$ are related under the mild assumption of a "local approximate unit," i.e., when $x \in [\mathcal{A}x]$ for every $x \in \mathcal{H}$.

Proposition 6. *Let $\mathcal{L}$ be a hyperatomic CSL and let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be an algebra such that $\operatorname{Lat}\mathcal{A} = \mathcal{L}$ and such that $x \in [\mathcal{A}x]$, for every $x \in \mathcal{H}$. The following statements are equivalent.*

1. *$\mathcal{A}$ is $\operatorname{Alg}\mathcal{L}$-transitive.*
2. *Every vector $x \in \mathcal{H}$ is closed for $\mathcal{A}$.*
3. *Every invariant manifold for $\mathcal{A}$ is closed.*

Moreover, when any of these conditions hold, every invariant manifold for $\mathcal{A}$ is singly generated; i.e., if $\mathcal{M} \in \operatorname{Man}(\mathcal{A})$, then there exists $x \in \mathcal{H}$ such that $\mathcal{M} = \{\mathcal{A}x\}$.

Before beginning the proof, we remark that while every element of $\mathcal{L}$ is singly generated as an $\operatorname{Alg}\mathcal{L}$ module, it is not *a priori* clear that every element of $\mathcal{L}$ is singly generated as an $\mathcal{A}$ module.

Proof. ($1 \Rightarrow 2$) Let $x \in \mathcal{H}$. Since $\mathcal{L}$ is hyperatomic, Theorem 4 shows that $[\operatorname{Alg}\mathcal{L}x] = \{\operatorname{Alg}\mathcal{L}x\}$ which, by assumption, is $\{\mathcal{A}x\}$. Thus statement (2) holds.

($2 \Rightarrow 1$) By assumption, for all x, we have $\{\mathcal{A}x\} = [\mathcal{A}x] \in \mathcal{L}$. Since $\{\operatorname{Alg}\mathcal{L}x\} = [\operatorname{Alg}\mathcal{L}x]$ is the smallest element of $\mathcal{L}$ which contains x and $x \in \{\mathcal{A}x\}$ by hypothesis, we see $\{\operatorname{Alg}\mathcal{L}x\} \subseteq \{\mathcal{A}x\} \subseteq \{\operatorname{Alg}\mathcal{L}x\}$. Thus, $\{\mathcal{A}x\} = \{\operatorname{Alg}\mathcal{L}x\}$, for all $x \in \mathcal{H}$.

($1 \Rightarrow 3$) Suppose that $x_1, x_2 \in \mathcal{H}$. We claim that there is a vector x in $\mathcal{H}$ such that

$$\{\mathcal{A}x\} = \{\mathcal{A}x_1\} \vee \{\mathcal{A}x_2\} \tag{2}$$

Write P_1 and P_2 for the projections onto $\{\operatorname{Alg}\mathcal{L}x_1\} = \{\mathcal{A}x_1\}$ and $\{\operatorname{Alg}\mathcal{L}x_2\} = \{\mathcal{A}x_2\}$. These projections are in $\mathcal{L}$ and hence in $\operatorname{Alg}\mathcal{L}$. Let $x = x_1 + P_1^{\perp}x_2$. We will show that $\{\operatorname{Alg}\mathcal{L}x\} = \{\operatorname{Alg}\mathcal{L}x_1\} \vee \{\operatorname{Alg}\mathcal{L}x_2\}$.

Since $x_1 = P_1x$, it follows immediately that $\{\operatorname{Alg}\mathcal{L}x_1\} \subseteq \{\operatorname{Alg}\mathcal{L}x\}$. Now let $y \in \{\operatorname{Alg}\mathcal{L}x_2\} \cap \{\operatorname{Alg}\mathcal{L}x_1\}^{\perp}$. Then there is $T \in \operatorname{Alg}\mathcal{L}$ such that $y = Tx_2$. Since $x_2 = P_1^{\perp}x_2 + P_1x_2$, we have

$$y = Tx_2 = TP_1^{\perp}x_2 + TP_1x_2 = TP_1^{\perp}x_2 + P_1TP_1x_2.$$

Since $P_1^{\perp}y = y$,

$$y = P_1^{\perp}TP_1^{\perp}x_2 = P_1^{\perp}TP_1^{\perp}(x_1 + P_1^{\perp}x_2) = P_1^{\perp}TP_1^{\perp}x.$$

This shows that $\{\operatorname{Alg}\mathcal{L}x_2\} \cap \{\operatorname{Alg}\mathcal{L}x_1\}^{\perp} \subseteq \{\operatorname{Alg}\mathcal{L}x\}$. Combining this with $\{\operatorname{Alg}\mathcal{L}x_1\} \subseteq \{\operatorname{Alg}\mathcal{L}x\}$ gives $\{\operatorname{Alg}\mathcal{L}x_1\} \vee \{\operatorname{Alg}\mathcal{L}x_2\} \subseteq \{\operatorname{Alg}\mathcal{L}x\}$. The reverse inequality follows from the fact that $x = x_1 + P_1^{\perp}x_2$ and the claim is verified.

Now let $\mathcal{M}$ be an arbitrary invariant linear manifold for $\mathcal{A}$. We need to show that $\mathcal{M}$ is closed. Let Q be the projection onto the closure of $\mathcal{M}$. Clearly $Q \in \mathcal{L}$ and hence is a hyperatomic projection. Let $A_1, \dots, A_n$ be a family of atoms of $\mathcal{L}$ such that $Q =$

$E(A_1, \dots, A_n)$. Then $Q = E(A_1) \vee E(A_2) \vee \cdots \vee E(A_n)$. Notice that if x_j is a non-zero vector in $A_j\mathcal{H}$, then $[\operatorname{Alg}\mathcal{L}x_j] = \{\operatorname{Alg}\mathcal{L}x_j\} = \{\mathcal{A}x_j\} = E(A_j)$. Inductively applying (2), we see that there exists $y \in \mathcal{H}$ such that $\mathcal{M} = \{\mathcal{A}y\}$, whence $\mathcal{M}$ is singly generated. Since $\mathcal{A}$ is $\operatorname{Alg}\mathcal{L}$-transitive, y is a closed vector, whence $\mathcal{M}$ is closed.

($3 \Rightarrow 2$) Obvious.

It remains to show that when $\operatorname{Man}(\mathcal{A}) = \mathcal{L}$, then every invariant manifold for $\mathcal{A}$ is singly generated. Let $\mathcal{M}$ be an invariant linear manifold for $\mathcal{A}$. Then by hypothesis, the orthogonal projection Q onto $\mathcal{M}$ belongs to $\mathcal{L}$, hence there is a vector $x \in \mathcal{H}$ such that $\mathcal{M} = \{\operatorname{Alg}\mathcal{L}x\}$. Clearly $x \in \mathcal{M}$. Now $x \in [\mathcal{A}x] = \{\mathcal{A}x\}$ and since $\{\operatorname{Alg}\mathcal{L}x\}$ is the smallest element of $\mathcal{L}$ containing x, we conclude that $\{\mathcal{A}x\} \supseteq \{\operatorname{Alg}\mathcal{L}x\} \supseteq \{\mathcal{A}x\}$. ∎

The following theorem requires the Kadison transitivity theorem for its proof and is (partially) an extension of that theorem.

Theorem 7. *Let $\mathcal{L}$ be a hyperatomic CSL. Suppose that $\mathcal{A} \subseteq \operatorname{Alg}\mathcal{L}$ is a norm closed operator algebra such that $\overline{\mathcal{A}}^{wot} = \operatorname{Alg}\mathcal{L}$. Assume that $E\mathcal{A}F \subseteq \mathcal{A}$ for all atoms E and F of $\mathcal{L}$ and that $E\mathcal{A}E$ is a C*-algebra for each atom. Then* $\operatorname{Man}(\mathcal{A}) = \operatorname{Lat}(\mathcal{A}) = \mathcal{L}$*, every element of* $\operatorname{Man}(\mathcal{A})$ *is singly generated, and $\mathcal{A}$ is* $\operatorname{Alg}\mathcal{L}$*-transitive.*

Proof. Observe that $E\mathcal{A}E$ is a C*-algebra which is weakly dense in $E\operatorname{Alg}\mathcal{L}\,E = \mathcal{B}(E\mathcal{H})$; thus $E\mathcal{A}E$ is an irreducible C*–subalgebra of $\mathcal{B}(E\mathcal{H})$.

We first assume that the identity operator I is generated by a single atom E_0 of $\mathcal{L}$. We shall prove that the invariant manifold for $\mathcal{A}$ generated by a unit vector in $E_0\mathcal{H}$ is all of $\mathcal{H}$. So fix a unit vector $\xi \in E_0\mathcal{H}$ and let $x \in \mathcal{H}$ be any vector. Let $\{Q_n\}_{n=0}^{\infty}$ be a sequence of projections in $\mathcal{L}''$ such that each Q_n is a finite sum of atoms of $\mathcal{L}$, $E_0 \le Q_0$, $\sum_{n=0}^{\infty} Q_n = I$ and $\sum_{n=1}^{\infty} \|Q_n x\| < \infty$.

Fix $n \ge 0$. Write $Q_n = \sum_{j=1}^{k_n} E_{n,j}$ as a finite sum of atoms of $\mathcal{L}$ and let $x_n = Q_n x$. Since $E_{n,j}\operatorname{Alg}\mathcal{L}\,E_0 = \mathcal{B}(E_0\mathcal{H}, E_{n,j}\mathcal{H})$, and $E_{n,j}\mathcal{A}E_0$ is weakly dense in $E_{n,j}\operatorname{Alg}\mathcal{L}\,E_0$, we may find a norm one operator $Y_{n,j} \in \mathcal{A}$ such that $Y_{n,j} = E_{n,j}Y_{n,j}E_0$. Hence we may find a unit vector $u_{n,j} \in E_0\mathcal{H}$ such that $\|Y_{n,j}u_{n,j}\| > 1/2$. By Kadison's transitivity theorem, there exist unitary operators $Z_{n,j} \in E_{n,j}\mathcal{A}E_{n,j}$ and $W_{n,j} \in E_0\mathcal{A}E_0$ such that

$$\frac{\|E_{n,j}x_n\|}{\|Y_{n,j}u_{n,j}\|} Z_{n,j}Y_{n,j}u_{n,j} = E_{n,j}x_n \quad \text{and} \quad u_{n,j} = W_{n,j}\xi.$$

Writing

$$A_{n,j} = \frac{\|E_{n,j}x_n\|}{\|Y_{n,j}u_{n,j}\|} Z_{n,j}Y_{n,j}W_{n,j},$$

we see that

$$A_{n,j} \in \mathcal{A}, \quad \|A_{n,j}\| < 2\|E_{n,j}x_n\|, \quad \text{and} \quad A_{n,j}\xi = E_{n,j}x_n.$$

Therefore, if $B_n = \sum_{j=1}^{k_n} A_{n,j}$, we find $B_n \in \mathcal{A}$ and $B_n\xi = x_n$. Moreover, since $E_{n,j}A_{n,j} = A_{n,j}$, we find that for any $\eta \in \mathcal{H}$, $\|B_n\eta\|^2 = \sum_{j=1}^{k_n} \|E_{n,j}A_{n,j}\eta\|^2$, so

$$\|B_n\| \le \left\{ \sum_{j=1}^{k_n} \|A_{n,j}\|^2 \right\}^{1/2} < 2\|x_n\|.$$

Notice also that $B_n = B_n E_0$ by construction.

The fact that $\sum_{n=1}^{\infty} \|x_n\| < \infty$ shows that the series $\sum_{n=0}^{\infty} B_n$ converges uniformly to an element $B \in \mathcal{A}$. Clearly, $B\xi = \sum_{n=0}^{\infty} B_n \xi = \sum_{n=0}^{\infty} x_n = x$. Thus we have shown that the invariant manifold generated by ξ is all of $\mathcal{H}$.

Furthermore, notice that our construction shows the following:

a) $B = BE_0$;
b) if E is an atom of $\mathcal{L}$ such that $Ex = 0$, then $EB = 0$; and
c) $B = \lim_{n\to\infty} R_n$, where for each n, $R_n = \sum_{j=1}^{p_n} C_{n,j}$ is a finite sum of elements $C_{n,j} \in \mathcal{A}$ which satisfy $C_{n,j} = E_{n,j} C_{n,j} F_{n,j}$ for some atoms $E_{n,j}$ and $F_{n,j}$ of $\mathcal{L}$.

Returning to the general case, if E is any atom from $\mathcal{L}$, we may compress to $P(E)$ (i.e., replace $\mathcal{A}$ by $P(E)\mathcal{A}P(E)$ acting on $P(E)\mathcal{H}$) and apply the argument above to obtain the following: if ξ is any non-zero vector in $E\mathcal{H}$ and if $x \in P(E)\mathcal{H}$, then there is $B \in \mathcal{A}$ such that $B\xi = x$, $B = BE$, and $Fx = 0$ implies $FB = 0$ for all atoms F. (There is one delicate point: our hypotheses do not guarantee that $P(E)\mathcal{A}P(E) \subseteq \mathcal{A}$, but in the construction of B, B is a norm limit of elements which are finite sums of elements of the form $F_1 X F_2$ with F_2 and F_2 atoms of $\mathcal{A}$. Such elements are in $\mathcal{A}$, by our hypotheses.)

Now let $\mathcal{M}$ be an invariant linear manifold under $\mathcal{A}$. Let P be the projection onto $\overline{\mathcal{M}}$. Then P is invariant under $\mathcal{A}$ and, hence, under $\operatorname{Alg}\mathcal{L}$. So, $P \in \mathcal{L}$.

Let $E_1, \dots, E_n$ be independent atoms which generate P. So, $P = P(E_1, \dots E_n)$ and $E_i \operatorname{Alg}\mathcal{L}\, E_j = 0$ whenever $i \neq j$. There is a vector $x \in \mathcal{M}$ such that $E_i x \neq 0$ for all i. (Let $y_i \in E_i\mathcal{H}$ with $\|y_i\| = 1$ and approximate $\sum y_i$ in norm by an element of $\mathcal{M}$.)

Clearly $\mathcal{A}x \subseteq \mathcal{M}$. We will prove that $P\mathcal{H} \subseteq \mathcal{A}x$; this implies that $\mathcal{M} = P\mathcal{H}$, whence $\mathcal{M}$ is closed and singly generated. Let $y \in P\mathcal{H}$ be arbitrary. Write $y = y_1 + \cdots + y_n$, where $y_i \in P(E_i)\mathcal{H}$ for each i. This can be done since $P = \bigvee_i P(E_i)$.

For each i, there is an element $B_i \in \mathcal{A}$ such that $B_i x_i = y_i$, $B_i = B_i E_i$, and $F y_i = 0$ implies $F B_i = 0$, for all atoms F. Let $B = B_1 + \cdots + B_n$. Then

$$\begin{aligned} Bx &= B_1 x + \cdots + B_n x \\ &= B_1 E_1 x + \cdots + B_n E_n x \\ &= B_1 x_1 + \cdots + B_n x_n \\ &= y_1 + \cdots + y_n = y. \end{aligned}$$

Thus, $y \in \mathcal{A}\mathcal{H}$ and $P\mathcal{H} \subseteq \mathcal{A}x$.

Finally, since $\mathcal{A}$ is weakly dense in $\operatorname{Alg}\mathcal{L}$, for every $y \in \mathcal{H}$ we have $\{\mathcal{A}y\} = [\mathcal{A}y] = [\operatorname{Alg}\mathcal{L}y]$. Since $\mathcal{L}$ is hyperatomic, we have $\{\operatorname{Alg}\mathcal{L}y\} = [\operatorname{Alg}\mathcal{L}y]$, so $\{\mathcal{A}y\} = \{\operatorname{Alg}\mathcal{L}y\}$ for every $y \in \mathcal{H}$. It follows that $\mathcal{A}$ is $\operatorname{Alg}\mathcal{L}$-transitive. $\square$

This theorem implies immediately a result tacit in the proof of the automatic closure theorem in [13]:

Corollary 8. *Let $\mathcal{K}$ be the algebra of compact operators and suppose $\mathcal{L}$ is a hyperatomic CSL. Then every invariant linear manifold for $\mathcal{K} \cap \operatorname{Alg}\mathcal{L}$ is closed.*

2. TAF algebras

We turn now to representations of strongly maximal triangular AF (TAF) algebras. These are subalgebras of AF C*-algebras arising as limits of triangular digraph algebras and have been extensively studied; see, for example, [26, 23, 14, 27, 7]. If $\mathcal{A}$ is a closed subalgebra of

an AF C*-algebra $\mathcal{C}$, then $\mathcal{A}$ is *triangular AF* or *TAF* if $\mathcal{A} \cap \mathcal{A}^*$ is a canonical masa in $\mathcal{C}$. A masa $\mathcal{D}$ is a *canonical masa* in $\mathcal{C}$ if the closed span of $N_{\mathcal{D}}(\mathcal{C})$ is $\mathcal{C}$, where

$$N_{\mathcal{D}}(\mathcal{C}) = \big\{ f \in \mathcal{C} : f \text{ is a partial isometry, } fdf^*, f^*df \in \mathcal{D} \text{ for } d \in \mathcal{D} \big\}.$$

A triangular algebra $\mathcal{A}$ is *strongly maximal* if $\overline{\mathcal{A} + \mathcal{A}^*} = \mathcal{C}$.

Let $\mathcal{A}$ be a strongly maximal triangular AF subalgebra of the AF C*-algebra $\mathcal{C}$ with $\mathcal{D} = \mathcal{A} \cap \mathcal{A}^*$ a canonical masa in $\mathcal{C}$. For reasons we will explain momentarily, we consider representations $\pi \colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ satisfying the following conditions:

1. π is the restriction to $\mathcal{A}$ of a $*$-representation ρ of $\mathcal{C}$ on $\mathcal{H}$;
2. $\pi(\mathcal{D})$ is weakly dense in a masa in $\mathcal{B}(\mathcal{H})$; and
3. $\mathrm{Lat}(\pi(\mathcal{A}))$ has order type $-\mathbb{N}$ and is multiplicity free.

Representations satisfying the first two conditions are called *masa preserving* [21, p. 130]. Since $\mathrm{Lat}(\rho(\mathcal{C})) \subseteq \mathrm{Lat}(\pi(\mathcal{A})) \cap \mathrm{Lat}(\pi(\mathcal{A}^*)) = \{0, I\}$, the $*$-representation ρ is necessarily irreducible. We will occasionally call a representation which satisfies conditions 1, 2, and 3 an *admissible representation*.

If $\mathcal{A}$ is non-unital, so is $\mathcal{C}$. By $\mathcal{A}^+$ we mean the obvious subalgebra of the unitization $\mathcal{C}^+$ of $\mathcal{C}$, and it is easy to see that $\mathcal{A}^+$ is a strongly maximal TAF subalgebra of $\mathcal{C}^+$. Since $\mathrm{Man}\,\pi(\mathcal{A}) = \mathrm{Man}(\pi(\mathcal{A}^+))$, we lose no generality by assuming that all algebras and representations are unital, and thus we make this assumption in the sequel.

The simplest example of a representation satisfying the three conditions above is the Smith representation of the standard embedding algebra acting on $\ell^2(-\mathbb{N})$ [21, Example I.2]. In fact, for standard embedding algebras, [21, Theorem III.2.1] shows that $\mathrm{Lat}(\pi(\mathcal{A}))$ is multiplicity free for representations π satisfying all the other conditions above.

A more general class of strongly maximal TAF algebras, the $\mathbb{Z}$-analytic algebras considered in [21, 24, 25], also admit representations of this form. However, not all strongly maximal TAF algebras have representations satisfying conditions 1, 2, and 3; for example, the refinement embedding algebras (see [23, 21]) have no such representations. Further, for a masa preserving representation of a refinement embedding algebra, there is a non-closed invariant linear manifold.

We have previously observed that the second condition implies that $\pi(\mathcal{A})$ is σ-weakly dense in the CSL algebra $\mathrm{Alg}\,\mathrm{Lat}(\pi(\mathcal{A}))$. However, since $\mathcal{A}$ is a strongly maximal TAF algebra, more is true: [21, Proposition 0.1] shows that the second condition implies that $\mathrm{Lat}(\pi(\mathcal{A}))$ is a nest. (A nest is a totally ordered CSL.) Moreover, for many of the examples in [21], $\mathrm{Alg}\,\mathrm{Lat}(\pi(\mathcal{A}))$ is multiplicity free. If every invariant manifold for $\pi(\mathcal{A})$ is closed, then necessarily the nest $\mathrm{Lat}(\pi(\mathcal{A}))$ is hyperatomic.

Furthermore, if $\mathcal{A}$ is a $\mathbb{Z}$-analytic subalgebra of a simple AF C*–algebra and if π is an irreducible representation of $C^*(\mathcal{A})$ which satisfies condition 2, then by [21, Proposition III.3.2] $\mathrm{Lat}\,\pi(\mathcal{A})$ is a nest whose order type is a subset of the integers. Since a nest is hyperatomic if, and only if, the complementary nest is well-ordered, the hyperatomic nests with order type a subset of the integers are just the finite nests and the nests of order type $-\mathbb{N}$. Automatic closure for invariant manifolds is trivial when $\mathrm{Lat}\,\pi(\mathcal{A})$ is a multiplicity free finite nest. If the nest is finite but not multiplicity free (the nest may be the trivial nest $\{0, I\}$, for example), the automatic closure question is open. For nests with order type $-\mathbb{N}$, Theorem 10 below gives an affirmative answer to the question.

Some motivation for, in effect, restricting to irreducible representations can be found in the following fact, although it does not reduce the study of masa preserving representations to the study of irreducible masa preserving representations.

Lemma 9. *Let π be a representation of $\mathcal{C}$ such that $\pi(\mathcal{D})$ is weakly dense in a masa in $\mathcal{B}(\mathcal{H})$. Every invariant linear manifold for $\pi(\mathcal{A})$ is closed if, and only if, π decomposes as a direct sum of finitely many irreducible representations $\pi = \oplus_{i=1}^{n} \pi_i$ and each invariant linear manifold for $\pi_i(\mathcal{A})$ is closed $(i = 1, \dots n)$.*

Proof. If $\mathrm{Man}(\pi(\mathcal{A})) = \mathrm{Lat}(\pi(\mathcal{A}))$, then since every invariant manifold for $\pi(\mathcal{C})$ is also an invariant manifold for $\pi(\mathcal{A})$, the discussion in the introduction shows that π decomposes as required. Then every linear manifold invariant for $\pi_i(\mathcal{A})$ is also invariant for $\pi(\mathcal{A})$. Conversely, since $\pi(\mathcal{D})''$ is a masa in $\mathcal{B}(\mathcal{H})$, every invariant manifold $\mathcal{M}$ for $\pi(\mathcal{A})$ decomposes as finite orthogonal sum of invariant manifolds for $\pi_i(\mathcal{A})$, whence $\mathcal{M}$ is closed. □

We can now state the main theorem of this section.

Theorem 10. *Let $\mathcal{A}$ be a strongly maximal triangular subalgebra of an AF C*-algebra $\mathcal{C}$. If $\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a masa preserving, order type $-\mathbb{N}$, multiplicity free representation, then every invariant linear manifold for $\pi(\mathcal{A})$ is closed and singly generated.*

While the proofs of Theorem 10 and of Theorem 7 employ similar methods, this theorem is not subsumed by Theorem 7 as $\pi(\mathcal{A})$ is not a bimodule over the algebra generated by the atoms of the nest.

To prove Theorem 10, we need to describe admissible representations in terms of coordinates. The full development of such coordinates is technical, and the reader is referred to [19, 22, 29] for more complete treatments. Associated to each AF C*-algebra $\mathcal{C}$ there is a unique AF groupoid $\mathfrak{G}$ so that the C*-algebra of $\mathfrak{G}$, $C^*(\mathfrak{G})$, and $\mathcal{C}$ are isomorphic as C*-algebras. The elements of $C^*(\mathfrak{G})$ can be identified with continuous functions on $\mathfrak{G}$. With this identification, $C(\mathfrak{G}_0)$ embeds in $C^*(\mathfrak{G})$ and is a canonical masa in $C^*(\mathfrak{G})$. In particular, we may identify $\mathcal{D}$ with $C(\mathfrak{G}_0)$ and $\mathcal{C}$ with $C^*(\mathfrak{G})$. Given a unit $e \in \mathfrak{G}_0$, its orbit is the set

$$[e] := \{f \in \mathfrak{G}_0 : \text{ for some } x \in \mathfrak{G}, e = x^{-1}x \text{ and } xx^{-1} = f\}.$$

Given a triangular algebra $\mathcal{A}$ with $\mathcal{D} \subseteq \mathcal{A} \subseteq \mathcal{C}$, there is an anti-symmetric subset of $\mathfrak{G}$ containing $\mathfrak{G}_0$, denoted $\mathrm{Spec}(\mathcal{A})$, so that $\mathcal{A}$ is isomorphic to $\{f \in C^*(\mathfrak{G}) : \operatorname{supp} f \subset \mathrm{Spec}(\mathcal{A})\}$. If $\mathcal{A}$ is strongly maximal, then $\mathrm{Spec}(\mathcal{A})$ totally orders each orbit in $\mathfrak{G}_0$. Similar coordinates can be defined for other groupoid C*-algebras; see [29, 16, 18].

By Theorem II.1.1 in [21], each representation satisfying conditions 1 and 2 is unitarily equivalent to a representation of the type constructed below. Recall that a *1-cocycle* is a groupoid homomorphism $\alpha\colon \mathfrak{G} \to G$, where G is an abelian group; for us G is $\mathbb{T} = \{x \in \mathbb{C} : |x| = 1\}$. As these are the only cocycles we consider, we abbreviate this to cocycle.

For $v \in N_{\mathcal{D}}(\mathcal{C})$, let σ_v denote the partial homeomorphism on $\mathfrak{G}_0 = \hat{D}$ induced by the map $d \mapsto vdv^*$. A measure μ on $\mathfrak{G}_0$ is *$\mathfrak{G}$-quasi-invariant* if, for each $v \in N_{\mathcal{D}}(\mathcal{C})$, the measures μ and $\mu \circ \sigma_v$ are equivalent, as measures on the domain of σ_v. Given such a measure μ, we say a cocycle $\alpha\colon \mathfrak{G} \to \mathbb{T}$ is *μ-measurable* if, for each $v \in N_{\mathcal{D}}(\mathcal{C})$, the function, denoted α_v, from domain of σ_v to $\mathbb{C}$ that sends x to $\alpha(x, \sigma_v(x))$ is measurable.

Since $\mathcal{C}$ is generated by the diagonal $D \cong C(\mathfrak{G}_0)$ and $N_D(\mathcal{C})$, we can build a representation ρ of $\mathcal{C}$ on $L^2(\mathfrak{G}_0, \mu)$ by defining the action of ρ on $\mathcal{D}$ and on $N_{\mathcal{D}}(\mathcal{C})$ and then extending by

linearity to $\mathcal{C}$. For $f \in \mathcal{D} \cong C(\mathfrak{G}_0)$ and $v \in N_{\mathcal{D}}(\mathcal{C})$, define respectively

$$\rho(f)\eta = f\eta, \qquad \rho(v)\eta = \alpha_v \left[\frac{d(\mu \circ \sigma_v)}{d\mu}\right]^{1/2} (\eta \circ \sigma_v).$$

Theorem 11. [21, Theorem II.1.1] *Every representation satisfying conditions 1 and 2 is unitarily equivalent to one arising as above from a $\mathfrak{G}$-quasi-invariant measure μ and a μ-measurable cocycle α, for some choice of μ and α.*

Suppose now that π is an admissible representation. Since π is multiplicity free, the support of μ is a countable set S. The irreducibility of π implies that S is the orbit of a single point of $\mathfrak{G}_0$ and because $\mathrm{Lat}(\pi(\mathcal{A}))$ has order type $-\mathbb{N}$, S is ordered by $\mathrm{Spec}(\mathcal{A})$ as $-\mathbb{N}$. Thus $L^2(\mathfrak{G}_0, \mu)$ may be identified with with $\ell^2(-\mathbb{N})$. Using $\{e_j : j \in -\mathbb{N}\}$ for the basis vectors of $\ell^2(-\mathbb{N})$ and letting P_n be the projection onto $\overline{\mathrm{span}\{e_k : k < n\}}$, the lattice of $\pi(\mathcal{A})$ is $\mathrm{Lat}\,\pi(\mathcal{A}) = \{0, I\} \cup \{P_n : n \in -\mathbb{N}\}$.

Given a finite subset $Y \subset \mathfrak{G}_0$, we associate a digraph algebra (an algebra isomorphic to a finite-dimensional CSL algebra) to $\mathfrak{S} = \mathrm{Spec}(\mathcal{A}) \cap (Y \times Y)$, namely the span of the rank one operators $e_x(e_y)^*$ for $(x, y) \in \mathfrak{S}$ acting on the space $\ell^2(\{e_y : y \in Y\})$.

Lemma 12. *Given a finite subset $Y \subset \mathfrak{G}_0$, let S be the digraph algebra associated to $\mathfrak{S} =$ $\mathrm{Spec}(\mathcal{A}) \cap (Y \times Y)$. There is an isometric inclusion $\zeta\colon S \to \mathcal{A}$ so that s is in the graph of $\zeta(e_s)$ for each $s \in \mathfrak{S}$.*

Lemma 12 was proved in [7, Lemma 4.2]; we need only observe that the inclusion constructed there is isometric.

Proof of Theorem 10. Let $\{P_n : n \in -\mathbb{N}\}$ be the projections onto the elements of $\mathrm{Lat}(\pi(\mathcal{A}))$, listed in decreasing order; thus $0 < \ldots < P_{-2} < P_{-1} < P_0 = I$. For $n \in \{-1, -2, \ldots\}$ we let e_n be a unit vector in the range of $P_{n+1} - P_n$; since $\mathrm{Lat}(\pi(\mathcal{A}))$ is multiplicity free, $\{e_n\}_{n=-\infty}^{-1}$ is an orthonormal basis for $\mathcal{H}$.

We first show that the singly generated invariant manifolds are closed. Consider the manifold M generated by e_{-1}. Clearly, M is dense in $\mathcal{H}$. We shall show that if $x \in \mathcal{H}$ and $\langle x, e_{-1}\rangle \neq 0$, then there exists $T \in \pi(\mathcal{A})$ such that $Te_{-1} = x$ and, moreover, that T can be taken so that $T^{-1} \in \pi(\mathcal{A})$.

Fix an element $x \in \mathcal{H}$ with $\langle x, e_{-1}\rangle = 1$ and choose a decreasing sequence of positive numbers ε_k such that $\sum_{k=1}^{\infty} \varepsilon_k = \delta$, where $\delta < (1 + \|x\|)^{-1}$. Since $P_{-n}^{\perp}$ are finite rank and converge strongly to I, we may choose an increasing sequence $n_k \in \mathbb{N}$ so that $\|P_{-n_1}x\| < 1$ and $\left\|P_{-n_k}x - P_{-n_{k+1}}x\right\| < \varepsilon_k$, for $k > 1$.

Let $x_1 = x - P_{-n_1}x$ and, for $k > 1$, let $x_k = P_{-n_k}x - P_{-n_{k+1}}x$. Clearly, $\sum_{k>1} \|x_k\| < \delta$. Since there is a natural identification between $(I - P_{-n_k})\mathcal{H}$ and $\mathbb{C}^{n_k}$, we may regard x_k as an element of $\mathbb{C}^{n_k}$.

Now let $X_1 \in M_{n_1}(\mathbb{C})$ be defined by $e_{-1}e_{-1}^* + x_1e_{-1}^* - I$. Here, e_{-1} denotes the "last standard basis vector" in $\mathbb{C}^{n_1}$. Since $\langle x, e_{-1}\rangle = 1$, we find that relative to the decomposition $I = (I - e_{-1}e_{-1}^*) + e_{-1}e_{-1}^*$, X_1 has the upper triangular form

$$X_1 = \begin{bmatrix} -I_{n_1-1} & v \\ 0 & 1 \end{bmatrix}.$$

Thus $X_1^2 = I_{n_1}$. For $k > 1$, let $X_k = x_ke_{-1}^* \in T_{n_k}(\mathbb{C})$.

Let $\mathfrak{O} \subset \mathfrak{G}_0$ be the support of the measure μ. Since $\mathfrak{O}$ has a natural identification with $-\mathbb{N}$, for each k, let $Y_k \subset \mathfrak{O}$ be that part identified with $\{-n_k, \dots, -2, -1\}$. Let $\zeta_n \colon T_{n_k}(\mathbb{C}) \to \mathcal{A}$ be the isometric embedding associated to Y_k given by Lemma 12. Since ζ_n is isometric and $\sum_{k>1} \|X_k\| = \sum_{k>1} \|x_k\| < \delta$, we see that the sum

$$\sum_{k=1}^{\infty} \zeta_{n_k}(X_k)$$

converges uniformly to an element $X \in \mathcal{A}$. Notice also that if we let $Z = \sum_{k>1} \zeta_{n_k}(X_k)$, then $X = \zeta_{n_1}(X_1) + Z$. Since $\zeta_{n_1}(X_1)$ is a square root of I, $\|\zeta_{n_1}(X_1)\| < 1 + \|x\|$, and $\|Z\| < \delta$, we find $X = \zeta_{n_1}(X_1)(I + \zeta_{n_1}(X_1)Z)$ is invertible and $X^{-1} \in \mathcal{A}$.

Let $T = \pi(X)$. Then T is an invertible element of $\pi(\mathcal{A})$ and an examination of the construction shows that $Te_{-1} = x$. Note that if $\langle x, e_{-1}\rangle = 0$, the same construction still gives an operator T in $\pi(\mathcal{A})$ such that $Te_{-1} = x$; in this case T is no longer invertible.

We conclude that if y_1 and y_2 are vectors in $\mathcal{H}$ with $\langle y_1, e_{-1}\rangle \neq 0$, then there exists $T \in \pi(\mathcal{A})$ such that $Ty_1 = y_2$. (Indeed, find $S_i \in \pi(\mathcal{A})$ such that $S_i e_{-1} = y_i$ and S_1 is invertible; then take $T = S_2 S_1^{-1}$.)

It follows from our work so far that if $x \in \mathcal{H}$ has $\langle x, e_{-1}\rangle \neq 0$, then the invariant manifold generated by x is $\mathcal{H}$, which is obviously closed.

Now let $x \in \mathcal{H}$ be an arbitrary unit vector and let M be the invariant manifold generated by x. The closure of M is an element of the nest, so $\overline{M} = P_{-n}$ for some n. Clearly, $\langle x, e_{-n}\rangle \neq 0$ and by "compressing" the argument above to P_{-n} we see that $M = P_{-n}$, so M is closed. Hence all singly generated invariant manifolds are closed.

The result now follows from Proposition 6. $\square$

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